

SINGULAR POINTS OF REAL QUARTIC CURVES VIA COMPUTER ALGEBRA

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Abstract

There are thirteen types of singular points for irreducible real quartic curves and seventeen types of singular points for reducible real quartic curves. This classification is originally due to D.A. Gudkov. There are nine types of singular points for irreducible complex quartic curves and ten types of singular points for reducible complex quartic curves. We derive the complete classification with proof by using the computer algebra system Maple. We clarify that the classification is based on computing just enough of the Puiseux expansion to separate the branches. Thus, the proof consists of a sequence of large symbolic computations that can be done nicely using Maple.

1 Introduction

The classification of singular points of real quartic curves is originally due to D.A. Gudkov [2,3,4,5]. He determined the individual types of singular points, as well as all possible sets of singular points that real quartic curves can have. In this paper, we will derive the thirteen individual types of singular points for irreducible real quartic curves and the seventeen individual types of singular points for reducible real quartic curves. Though our results are not new, the description of the equivalence relation is new and our proof is new and gives a very nice illustration of the role that computer algebra can play in doing proofs. Furthermore, our proof is self-contained and is the most elementary proof possible, which makes this material accessible to the widest possible audience.

The general question is how shall we classify singular points of real quartic curves. For each fixed degree n , we want a finite classification of singular points for all algebraic curves of degree n . Thus, in general, the local diffeomorphism type is not the desired criterion of classification for singular points. For irreducible quartic curves, there are only finitely many diffeomorphism types, but for reducible quartic curves, there are infinitely many equivalence classes with respect to local diffeomorphism. For example, in the Arnol'd notation, four lines intersecting at the origin represents an X_9 singular point which is really an infinite family of smoothly inequivalent singularities. Notice here that an irreducible real quintic curve can have an X_9 singular point. The tradition is

to treat these as one class by fiat. In our scheme the X_9 family will appear naturally as a single class.

Now let us describe how we will classify the individual types of singular points that a real quartic curve can have. Given any polynomial equation $F(x, y) = 0$, it is possible to solve for y in terms of x in the form of fractional power series, called Puiseux expansions. There is an algorithm for doing this, and the software Maple computes such Puiseux expansions, even for curves with literal coefficients. Our classification is based on taking just enough of the Puiseux expansions to separate the “branches,” and noting the exponents at which the “branches” separate. In other words, compute the Puiseux expansions to a power of x such that all expansions are unique. Then we will associate a tree-type graph, to which we will refer as a “tree diagram” or “diagram.” These diagrams will be described in detail below and will codify how the “branches” separate and will serve to classify the type of the singular point. At this point let us remark that the term “branch” already has a traditional meaning in this context. We are really interested in the distinct Puiseux expansions. In [7], C.T.C. Wall has coined the term “pro-branch” for the distinct Puiseux expansions.

In studying a singular point of an algebraic curve, the first thing to look at is the Newton polygon. (Our Newton polygons will follow the style of Walker [6].) Corresponding to each segment of the Newton polygon, there is a quasihomogeneous polynomial [p. 195,1]. If all such quasihomogeneous polynomials have no multiple factors, then the Newton polygon already tells us the type of the singularity. (Note that in this case, we know right away the exponents at which all of the Puiseux expansions separate.) But if there is a multiple factor, then it is necessary to examine the situation more closely. For this, we turn to the Puiseux expansions. As indicated above, the relevant definition on which our classification is based is new and appeals to the Puiseux expansions in an invariant way.

Let us note that we will classify the *real* singular points. (It is possible for a real quartic curve to have a complex conjugate pair of singular points. We will avoid this case.) By a simple translation of axes, we may assume that the singular point is at the origin. We will treat both irreducible and reducible curves, but note that the notions of irreducible and reducible are with respect to the complex numbers. Note also that we will not study reducible curves with multiple components.

The objects being classified are pairs whose first coordinate is a real quartic curve, specified by a 4th degree polynomial with real coefficients, considered up to a real nonzero multiplicative constant, and the second coordinate is a singular point of the curve in the first coordinate. Let the quartic curve be given by $f(x, y) = 0$, where

$$\begin{aligned}
f(x, y) = & a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy \\
& + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{40}x^4 + a_{31}x^3y \\
& + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4.
\end{aligned}$$

Since we may assume that our singular point is at the origin, we have $a_{00} = 0$. Since the point is singular, $a_{10} = a_{01} = 0$. In this paper we will use the term “tangent cone” to refer to the terms of lowest degree in $f(x, y)$. The degree of these terms is called the multiplicity of the point. If the point is of multiplicity four, then the curve must be reducible since any homogeneous polynomial of degree 4 must factor. Thus, for irreducible curves, we only need to study points of multiplicity three or two.

Let us now explain how all of the cases are enumerated. First we choose the tangent cone by choosing the tangent lines together with their multiplicities. The choice of tangent lines can be fixed by a linear change of coordinates. Moreover, by rotation of axes, we may assume no tangent line is vertical. For each tangent cone, we consider all possible Newton polygons. For each Newton polygon, we first consider the case where none of the quasihomogeneous polynomials corresponding to the segments of the Newton polygon have a multiple factor. Then we consider the cases where there is a multiple factor. When there is a multiple factor, the choice of this factor can be fixed by a linear change of coordinates. For irreducible quartic curves, the only case of this kind is the one with tangent cone $(y + x^2)^2$. In this case, Maple is used to compute the Puiseux expansions for the corresponding family of curves. The different types of singular points are then determined by the vanishing or nonvanishing of certain polynomials in the coefficients of this family; these polynomials are given to us by the Maple computations in the form of discriminant-like polynomial coefficients of the Puiseux expansions. The details of this are carried out in the next section.

To be more specific, the details of the following outline will be carried out in the next section. For irreducible quartic curves, by a linear change of coordinates, as described above, it suffices to consider the following families:

Multiplicity 3

$$y^3 + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0, \quad a \neq 0$$

$$y^2(y - x) + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0, \quad a \neq 0$$

$$y(y - x)(y - 2x) + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0, \quad a \neq 0$$

$$y(y^2 + x^2) + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0, \quad a \neq 0$$

Multiplicity 2

$$y^2 - x^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^4 + fx^3y + gx^2y^2 + hxy^3 + jy^4 = 0$$

$$y^2 + x^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^4 + fx^3y + gx^2y^2 + hxy^3 + jy^4 = 0$$

$$y^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^4 + fx^3y + gx^2y^2 + hxy^3 + jy^4 = 0, \quad a \neq 0$$

$$(y + x^2)(y - x^2) + ax^3y + bxy^2 + cx^2y^2 + dy^3 + exy^3 + fy^4 = 0$$

$$y^2 + x^4 + ax^3y + bxy^2 + cx^2y^2 + dy^3 + exy^3 + fy^4 = 0$$

$$(y + x^2)^2 + ax^3y + bxy^2 + cx^2y^2 + dy^3 + exy^3 + fy^4 = 0$$

Maple computation is needed only for the last family above. An interesting feature of the proof occurs at the end of this computation, where we show that every curve in the family

$$(y + x^2)^2 + bx^3y + bxy^2 + (1/4b^2 + d)x^2y^2 + dy^3 + 1/2bdxy^3 + fy^4 = 0$$

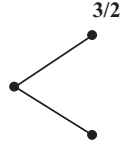
is reducible. This is the key to establishing that the list of double points is complete.

For reducible quartic curves, Maple computation is used to examine the cases where an irreducible cubic is tangent to the line component and where two conics share a common tangent. (The other cases are enumerated by mathematical common sense.)

Given an algebraic curve with a singular point at the origin, let us now describe how to associate a tree diagram to this singular point once we have the Puiseux expansions. Each time at least one “branch” separates, record the exponent where that happens. Place all such exponents in a row at the top. For each exponent in the top row, there corresponds a column of vertices. Each Puiseux expansion corresponds to exactly one vertex in that column, and those expansions with the same coefficients up to that exponent correspond to the same vertex. Braces will join those pairs of vertices, within a given column, that correspond to complex conjugate coefficients. In such a case, the only real solution of the original equation, satisfying the pair of expansions indicated by the braces, in a small enough neighborhood of the origin is (0,0).

In [7] C.T.C. Wall uses the term “pro-branches” to refer to the distinct Puiseux expansions belonging to a given singular point, and then defines a notion of *exponent of contact* between two pro-branches. It follows from Lemma 4.1.1 on page 68 of [7], that the diagram we assign to a singular point is invariant under a linear change of coordinates.

Example. $y^2 = -x^3$. Notice that $y = \pm i x^{3/2}$, which can also be written as $y = \pm(-x)^{3/2}$. For each $x < 0$, there are two distinct real solutions for y . Hence, the diagram is as shown below (without braces!).



We start with one vertex on the left corresponding to the power zero. Line segments are drawn connecting the vertices from left to right, where each polygonal path from left to right corresponds to Puiseux expansions having the same set of coefficients up to a given exponent. The diagram stops at the first exponent where each vertex in that column corresponds to exactly one Puiseux expansion. The key point is that this tree diagram uniquely specifies the singularity type (up to permutations of vertices within columns) provided that no tangent line at the origin is vertical.

Example. $x^2y + x^4 + 2xy^2 + y^3 = 0$. If $B := x^2y + x^4 + 2xy^2 + y^3$, the Maple command `puiseux (B, x = 0, y, 3)` tells us that the Puiseux expansions begin as follows:

$$y = -x + x^{3/2} \quad (\text{branch\#1})$$

$$y = -x - x^{3/2} \quad (\text{branch\#2})$$

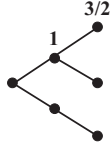
$$y = -x^2 \quad (\text{branch\#3})$$

In the next section, we will refer to the relevant truncated portion of the Puiseux expansion as the *Puiseux jet*. Notice that the coefficient of x in branch #1 and branch #2 is -1 , while the coefficient of x in branch #3 is 0 . So there is a splitting at the first power of x , which is indicated as



Next we must show the splitting of #1 from #2. Notice that the power of x at which #1 and #2 split is $3/2$.

Now our diagram looks like the following:



The diagram is now complete; notice that there are three distinct vertices in the column labeled $3/2$.

Definition of the equivalence relation: two singular points are equivalent if they have the same diagram as described above.

To summarize, we have described a precise procedure for assigning a diagram to a singular point of an algebraic curve and this assignment is invariant under a linear change of coordinates.

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Note: Due to diagram typesetting considerations, section 2 begins on the following page.

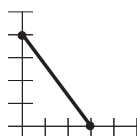
2 Classification and proof.

Irreducible curves.

Multiplicity 3.

Tangent cone: y^3 .

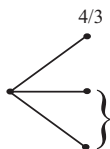
Newton polygon:



$$y^3 + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0, a \neq 0.$$

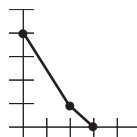
Puiseux jets : $y = -a^{1/3}x^{4/3}$ (three expansions here).

Diagram Type 1:



Tangent cone: $y^2(y - x)$.

Newton polygon:



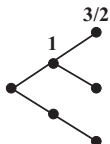
$$y^2(y - x) + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4, a \neq 0.$$

Puiseux jets (from Newton polygon; Maple not needed):

$$y = x$$

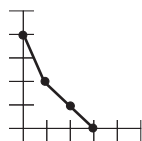
$$y = \pm\sqrt{a}x^{3/2}.$$

Diagram Type 2:



Tangent cone: $y(y-x)(y-2x)$.

Newton polygon:



$$y(y-x)(y-2x) + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0, a \neq 0.$$

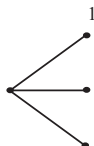
Puiseux jets:

$$y = 0$$

$$y = x$$

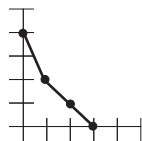
$$y = 2x.$$

Diagram Type 3:



Tangent cone: $y(y^2 + x^2)$.

Newton polygon:

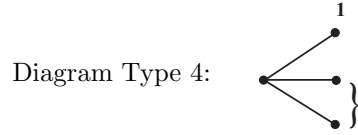


$$y(y^2 + x^2) + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0, a \neq 0.$$

Puiseux jets:

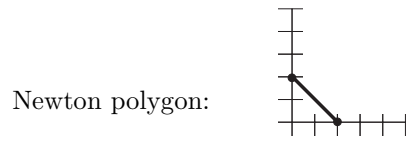
$$y = 0$$

$$y = \pm ix.$$



Multiplicity 2.

Tangent cone: $(y - x)(y + x) = y^2 - x^2$.

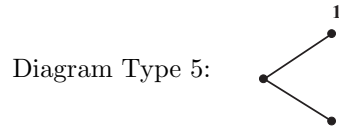


$$y^2 - x^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^4 + fx^3y + gx^2y^2 + hxy^3 + jy^4 = 0.$$

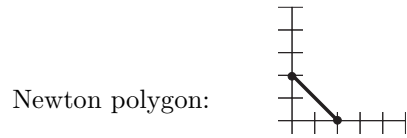
Puiseux jets:

$$y = x$$

$$y = -x.$$



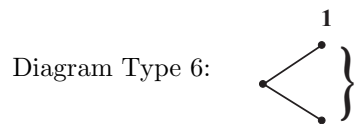
Tangent cone: $y^2 + x^2$.



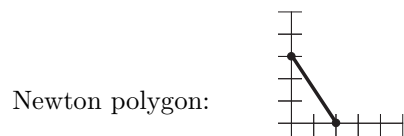
$$y^2 + x^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^4 + fx^3y + gx^2y^2 + hxy^3 + jy^4 = 0.$$

Puiseux jets:

$$y = \pm ix.$$

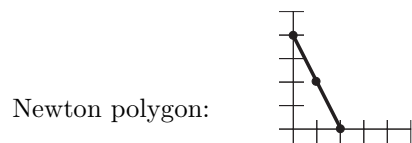
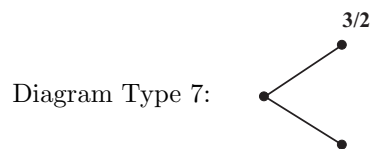


Tangent cone: y^2 .



$$y^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^4 + fx^3y + gx^2y^2 + hxy^3 + jy^4 = 0, a \neq 0.$$

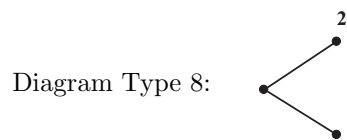
Puiseux jets:
 $y = \pm\sqrt{a} x^{3/2}$.



Quasihomogeneous factors: $(y + x^2)(y - x^2)$.

$$(y + x^2)(y - x^2) + ax^3y + bxy^2 + cx^2y^2 + dy^3 + exy^3 + fy^4 = 0.$$

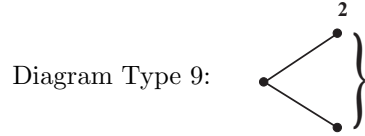
Puiseux jets:
 $y = x^2$
 $y = -x^2$.



Quasihomogeneous factors: $y^2 + x^4$.

$$y^2 + x^4 + ax^3y + bxy^2 + cx^2y^2 + dy^3 + exy^3 + fy^4 = 0.$$

Puiseux jets:
 $y = \pm ix^2$.



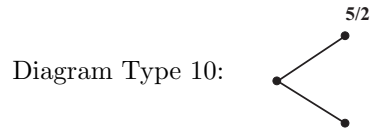
Quasihomogeneous factors: $(y + x^2)^2$.

$$A := (y + x^2)^2 + ax^3y + bxy^2 + cx^2y^2 + dy^3 + exy^3 + fy^4 = 0.$$

Notice that the quasihomogeneous polynomial $(y + x^2)^2$ has a double root. Thus, the family above contains several different types of singular points. We will determine polynomial conditions on the coefficients that will give all the different types of singular points by using Maple to compute a succession of Puiseux expansions. We begin by computing the Puiseux expansion of A using the Maple command `puiseux(A, x = 0, y, 0)`. Notice that the zero in the last argument instructs Maple to exhibit just enough of the Puiseux expansion to separate the “branches”!

Puiseux jets: $y = -x^2 + (a - b)^{1/2}x^{5/2}$.

Condition: $a \neq b$.



Case: $a = b$.

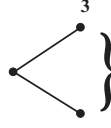
Puiseux jets:
 $y = -x^2 + x^3 \text{ Root Of } (-bZ + Z^2 + c - d).$

Condition: $c \neq \frac{1}{4}b^2 + d$.

Diagram Type 11. if $b^2 - 4(c - d) > 0$



Diagram Type 12. if $b^2 - 4(c - d) < 0$.



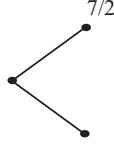
Case: $c = \frac{1}{4}b^2 + d$.

Puiseux jets:

$$y = -x^2 + (e - \frac{1}{2}bd)^{1/2}x^{7/2}.$$

Condition: $e \neq \frac{1}{2}bd$.

Diagram Type 13:



Case: $e = \frac{1}{2}bd$.

Notice that now the family has become

$$H := (y + x^2)^2 + bx^3y + bxy^2 + (\frac{1}{4}b^2 + d)x^2y^2 + dy^3 + \frac{1}{2}bdxy^3 + fy^4.$$

We cannot show that H is reducible by using the Maple command `factor(H)`. However, the Maple command `factor(H - fy^4)` shows that $H - fy^4 = \frac{1}{4}(2x^2 + 2y + bxy)(bxy + 2x^2 + 2dy^2 + 2y)$. Therefore, $H = \frac{1}{4}(2x^2 + bxy + 2y)^2 + \frac{d}{2}(2x^2 + bxy + 2y)y^2 + fy^4$, which is homogeneous in $(2x^2 + bxy + 2y)$ and y^2 , and thus factors.

This completes the classification of singular point types for irreducible real quartic curves.

Reducible Curves

Degrees of factors: 3, 1.

If the straight line does not pass through $(0,0)$, then there are three cases:

Diagram Type 1:

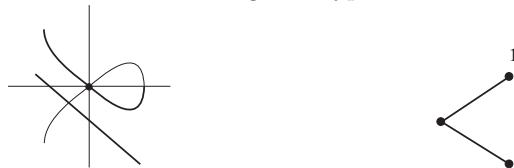


Diagram Type 2:

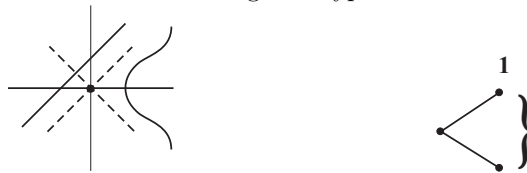
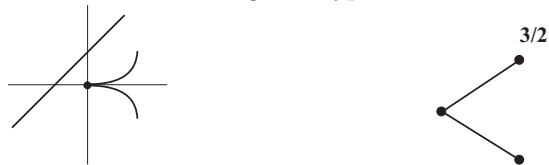


Diagram Type 3:



If the straight line does pass through $(0,0)$, then there are five cases:

Diagram Type 4:

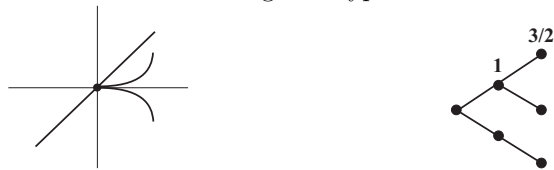


Diagram Type 5:

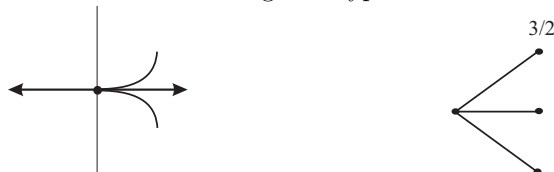


Diagram Type 6:



Consider the family $(y^2 - x^2 + ax^3 + bx^2y + cxy^2 + dy^3)(y - x) = 0$.

By using Maple, we obtain

Puiseux jets:

$$y = x + 0x^2$$

$$y = x + \frac{1}{2}(-a - b - c - d)x^2$$

$$y = -x.$$

Condition: $a + b + c + d \neq 0$.

Diagram Type 7:



Case: $a + b + c + d = 0$.

Then the cubic is reducible, so we are done.

Diagram Type 8:



Consider the family $(y - x + ax^2 + bxy + cy^2 + dx^3 + ex^2y + fxy^2 + gy^3)(y - x) = 0$.

By using Maple, we obtain

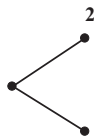
Puiseux jets:

$$y = x + 0x^2.$$

$$y = x + (-a - b - c)x^2.$$

Condition: $a + b + c \neq 0$

Diagram Type 9:



Case: $a = -b - c$.

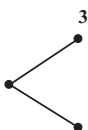
Puiseux jets:

$$y = x + 0x^3$$

$$y = x + (-f - g - e - d)x^3.$$

Condition: $f + g + e + d \neq 0$.

Diagram Type 10:



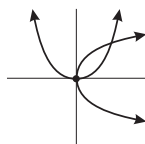
Case: $f + g + e + d = 0$.

Then the cubic is reducible, so we are done.

Degrees of factors: 2, 2.

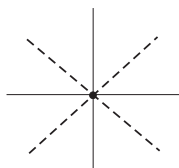
Case: Two distinct tangents at $(0, 0)$.

Diagram Type 1:



$$(y + ix + ix^2)(y - ix - ix^2) = y^2 + x^2 + 2x^3 + x^4 = y^2 + x^2(1 + x)^2.$$

Diagram Type 2:



Case: Two tangents coincide at $(0, 0)$.

$$(y + ax^2 + bxy + cy^2)(y + dx^2 + exy + fy^2) = 0.$$

Notice that $a \neq 0$ and $d \neq 0$. Calculation using Maple yields

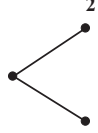
Puiseux jets:

$$y = -ax^2$$

$$y = -dx^2.$$

Condition: $a \neq d$.

Diagrams: Type 9.



Type 11.



Case: $a = d$.

Puiseux jets:

$$y = -dx^2 + edx^3$$

$$y = -dx^2 + bdx^3.$$

Condition: $b \neq e$.

Diagrams: Type 10.



Type 12.



Case: $b = e$.

Puiseux expansions:

$$y = -dx^2 + edx^3 + (-e^2d - cd^2)x^4$$

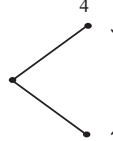
$$y = -dx^2 + edx^3 + (-e^2d - fd^2)x^4.$$

Condition: $c \neq f$.

Diagrams: Type 13.



Type 14.



If $c = f$, then we just have a curve of degree 2 with multiplicity 2.

Degrees of factors: 2, 1, 1.

or

Diagram Type 1:

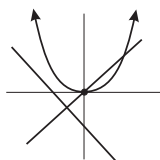
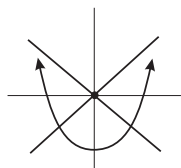


Diagram Type 2:

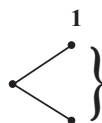
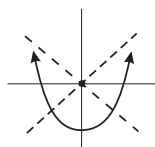


Diagram Type 9:

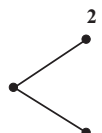
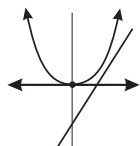


Diagram Type 6:

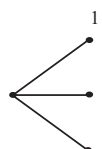
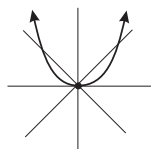


Diagram Type 8:

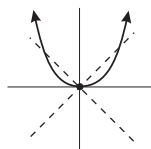
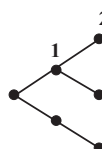
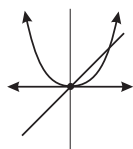


Diagram Type 7:



Degrees of factors: 1, 1, 1, 1.

Diagram Type 15:

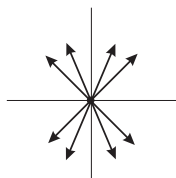


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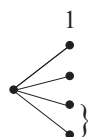
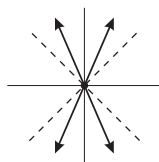
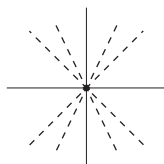


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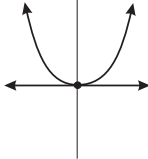
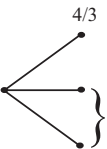
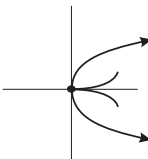
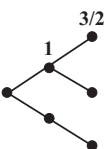
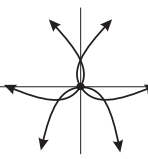
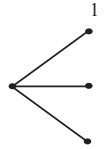
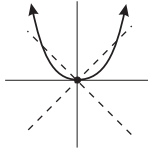
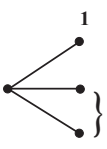
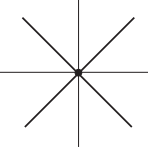
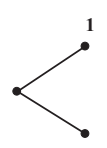
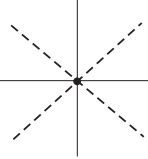



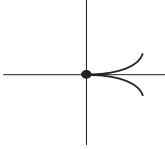
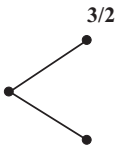
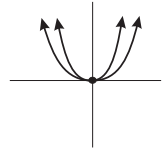
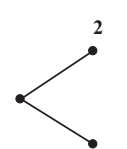
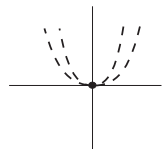
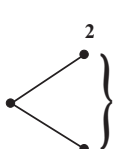
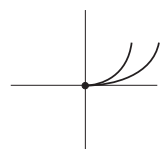
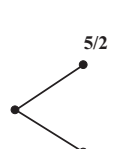
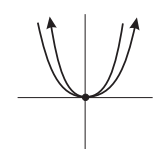
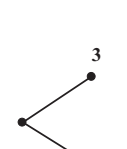
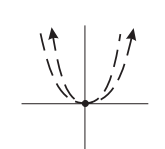
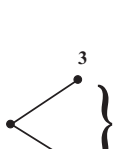
This completes the classification of singular point types for reducible real quartic curves.

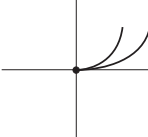
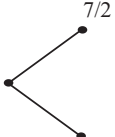
3 Summary of classification.

In this section, we summarize the classification by providing tables that show, for each singular point type, a simple example, together with a picture, the tree diagram, and the name of the singularity according to the Arnol'd notation. (Please note that in the table of reducible curves, the example will not always perfectly match the picture.)

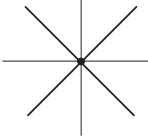

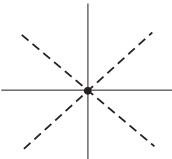
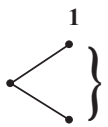
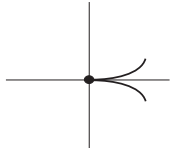
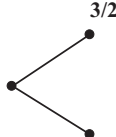
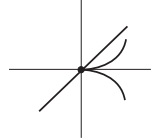
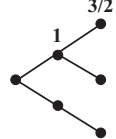
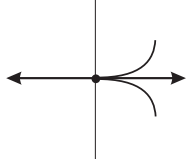
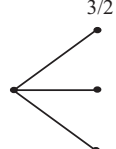
Irreducible Curves

Name	Picture	Diagram	Example
1. E_6			$y^3 - x^4 = 0$
2. D_5			$x^4 + xy^2 + y^3 = 0$
3. D_4			$x^4 + x^2y + xy^2 + y^3 = 0$
4. D_4^*			$x^4 + x^2y + y^3 = 0$
5. A_1			$y^2 - x^2 + x^4 = 0$
6. A_1^*			$y^2 + x^2 + x^4 = 0$

Name	Picture	Diagram	Example
7. A_2			$y^2 + x^3 + x^4 = 0$
8. A_3			$y^2 - x^4 + y^3 = 0$
9. A_3^*			$y^2 + x^4 + y^3 = 0$
10. A_4			$y^2 + 2x^2y + x^4 + x^3y = 0$
11. A_5			$y^2 + 2x^2y + x^4 + y^3 = 0$
12. A_5^*			$y^2 + 2x^2y + x^4 - y^3 = 0$

Name	Picture	Diagram	Example
13. A_6			$y^2 + 2x^2y + x^4 + x^2y^2 + \frac{1}{4}y^4 + y^3 = 0$

Reducible Curves

Name	Picture	Diagram	Example
1. A_1			$(y - 1)(y - 2)(y - x)(y + x) = 0$
2. A_1^*			$(y - 1)(y - 2)(x^2 + y^2) = 0$
3. A_2			$(y - 1)(y^2 - x^3) = 0$
4. D_5			$(y - x)(y^2 - x^3) = 0$
5. E_7			$y(y^2 - x^3) = 0$

Name	Picture	Diagram	Example
6. D_4			$(y - x)(y + x)(y - x^2) = 0$
7. D_6			$y(y - x)(y - x^2) = 0$
8. D_4^*			$(y - x)(y^2 + x^2 - x^3) = 0$
9. A_3			$(y - x^2)(y + x^2) = 0$
10. A_3^*			$y^2 + x^4 = 0$ (Reducible over \mathbb{C})
11. A_5			$(y + x^2)(y + x^2 + xy) = 0$

Name	Picture	Diagram	Example
12. A_5^*			$x^4 + 2x^2y + y^2x^2 + y^2 = 0$ (Reducible over \mathbb{C})
13. A_7			$(y + x^2 + xy)(y + x^2 + xy + y^2) = 0$
14. A_7^*			$x^4 + 2x^3y + 2x^2y + y^2x^2 + 2xy^2 + y^4 + y^2 = 0$ (Reducible over \mathbb{C})
15. X_9			$(y - x)(y + x)(y - 2x)(y + 2x) = 0$
16. X_9^*			$(y - x)(y + x)(x^2 + y^2) = 0$
17. X_9^{**}			$(x^2 + y^2)(x^2 + 4y^2) = 0$

References

- [1] E. Brieskorn and H. Knorrer, Plane algebraic curves, Birkhauser Verlag, Basel, 1986.
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- [6] R.J. Walker, Algebraic Curves, Princeton University Press, New Jersey, 1950, 2nd ed. published by Springer-Verlag, New York, 1978.
- [7] C.T.C. Wall, Singular points of plane curves, London Mathematical Society Student Texts 63, Cambridge University Press, Cambridge, 2004.

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